

Sub-Aging in a Domain Growth Model

L. Berthier

Laboratoire de Physique - ENS-Lyon and CNRS, F-69364, Lyon Cedex 07, France
(February 1, 2008)

We study analytically the aging dynamics of the $O(n)$ model in the limit of $n \rightarrow \infty$, with conserved and with non-conserved order parameter. While in the non-conserved dynamics, the autocorrelation function scales in the usual way $C(t, t_w) = \mathcal{C}(t/t_w)$, in the case of a conserved order parameter, ‘multiscaling’ manifests itself in the form $C(t, t_w) = \mathcal{C}(h(t)/h(t_w))$, with a relaxation time growing more slowly than the age of the system (sub-aging), and $h(t)$ a function growing faster than any length scale of the problem. In both cases, the effective temperature associated to the violation of the fluctuation theorem tends to infinity in the asymptotic limit of large waiting times.

PACS numbers: 05.70.Ln, 75.40.Gb, 82.20.Mj

LPENSL-TH-02/2000

I. INTRODUCTION

Aging of glassy systems is now well understood, at least from a qualitative point of view [1], and different approaches have been used to understand such a behaviour. One of them is the interpretation of aging in terms of a coarsening process. The picture is the following: consider for instance an Ising ferromagnet, which is quenched at time $t = 0$ below its critical temperature. When t increases, two types of domains emerge, with up and down spins. In the thermodynamic limit, equilibrium is never reached. At late times, domains have reached a typical size $L(t)$. It is thus natural to assume scaling laws for the different quantities of interest [2]. For instance, one can try the ansatz $S(\mathbf{k}, t) \sim L^d g(kL)$ for the structure factor (in a d -dimensional space), or $C(t, t_w) \sim F(L(t)/L(t_w))$ for the two-time autocorrelation function, where g and F are scaling functions. The growth law $L(t)$ determines then all the properties of the system. As an example, the droplet model for spin glasses [3] assumes a logarithmic growth, leading to $C(t, t_w) = F(\ln t / \ln t_w)$. If the growth law is given by $L(t) \sim t^\alpha$, like *e.g.* in a spinodal decomposition, one gets $C(t, t') = F(t/t_w)$. This last behaviour is called ‘simple aging’ and has been analytically shown to hold within various non-random models [1,4].

Moreover, the above functional form for the correlation function is also found analytically in some mean-field models of spin glasses, which give the general form for the correlation functions in the aging regime $C(t, t_w) = \mathcal{C}(\frac{h(t)}{h(t_w)})$, with h and \mathcal{C} two scaling functions [1] (valid in the two-time regime where both times are large, but with $1 < C < 0$). Although the notations are different, the functional form is the same as in coarsening processes, and it is then very natural to try to interpret the h -function as a relevant length scale for spin glasses, as was done for instance in ref. [5].

From the experimental and numerical side, it is found that a simple aging behaviour describes the data well, in many different systems. This is interpreted by saying that the relaxation time $t_r(t_w)$ of the system scales as the age t_w of the sample: $t_r \sim t_w$. However, a more subtle effect may appear, since t_r very often *grows more slowly than* t_w . This effect has been called *sub-aging*

[6]. In his pioneering experiments on polymer glasses, Struik [7] introduced the exponent μ from the relation $t_r \sim t_w^\mu$, with $\mu < 1$. Different values of μ have been reported: Struik used $\mu \sim 0.9$, experiments in spin glasses $\mu \sim 0.97$ [6], simulations of a structural glass were fitted using the value $\mu \sim 0.88$ [8], and recently, experiments on a gel gave $\mu \sim 0.9$ [9]. It can be checked (this point is discussed in detail in ref. [6]) that the μ -exponent is equivalent to the following choice of the h -function: $h(t) = \exp(\frac{1}{1-\mu}(\frac{t}{t_0})^{1-\mu})$. In accordance to what has been said above, this equivalence holds when $t_w \rightarrow \infty$ and $t - t_w \sim t_w^\mu$. Another function, the ‘enhanced power law’ form $h(t) = \exp(\ln^a(t/t_0))$ with $a > 1$, has been phenomenologically introduced in the context of spin glasses [6], and the value $a = 2.2$ was used to fit experiments. This in turn gives the relation $t_r \sim t_w / \ln^{a-1}(t_w)$, valid in the regime $t_w \rightarrow \infty$ and $t - t_w \sim t_w / \ln^{a-1}(t_w)$.

These choices are nonetheless not clearly motivated from a theoretical point of view, since the mean-field spin glass models discussed above only predict the existence of $h(t)$, and its analytical computation remains at present an open problem. In this context, simple models where h can be computed are much needed, but there are only few examples where sub-aging appears. Very recently, a model exhibiting sub-aging has been proposed by Rinn *et al* [10], who studied a slight variation of Bouchaud’s trap model for aging. This has given a theoretical support to the use of an exponent μ , even if its physical origin remains somewhat unclear. A scaling approach to the diffusion of a point particle in a low dimensional space has been proposed in ref. [11], and leads in some cases to a sub-aging which can be well described by an enhanced power law.

We study in the present paper a model for coarsening (the $O(n)$ model in the large- n limit) which also exhibits a sub-aging scaling in the autocorrelation function when the order parameter is not conserved. Its origin is the *simultaneous presence in the system of two different length scales*, whose consequence is the breakdown of the simple scaling laws generally used in domain growth processes. In particular, no t/t_w -scaling is found, and the relaxation time grows as $t_r \sim t_w / \sqrt{\ln t_w}$ (sub-aging). The autocorrelation is shown to be well represented in the asymptotic regime by an enhanced power law with $a = 3/2$, *i.e.* $h(t) = \exp((\ln x)^{3/2})$. Interestingly enough, $h(t)$ can not be interpreted in our example as a length scale. We do not want to argue that the model is a realistic one for the aging of polymers or spin glasses, but rather to give a possible physical explanation (the role of length scales [12]) for the absence of the ‘naive’ t/t_w -scaling, and exhibit a simple example where the h -function can be computed and discussed in terms of length scales, which has not been done so far.

II. THE $O(N)$ MODEL

This model is one of the few exactly solvable model for coarsening. It was first studied by Coniglio and Zannetti [13], who computed the scaling properties of the structure factor during the domain growth process. They pointed out the presence of the two mentioned length scales, and named ‘multiscaling’ the breakdown of the usual $S(\mathbf{k}, t) \sim L^d g(kL)$. Bray and Humayun have shown, however, that this multiscaling was a peculiarity of the large- n limit, and proved that for a large but finite value of n , a ‘normal scaling’ was recovered [14]. On an other hand, this ‘pathology’ has been shown to appear as a relevant preasymptotic effect in different coarsening models [15], like for instance the kinetic Ising model.

The model is defined through the Hamiltonian

$$H[\phi] = \int d^d \mathbf{x} \left(\frac{1}{2} (\nabla \phi)^2 + \frac{1}{4n} (n - \phi^2)^2 \right), \quad (2.1)$$

where $\phi(\mathbf{x}, t)$ is a n -component vector field in a d -dimensional space. Two different dynamics may be associated to this model, depending on whether or not the order parameter is conserved. In the case of a non-conserved order parameter, the dynamics is given by the so-called time dependent Ginzburg-Landau equation

$$\frac{\partial \phi(\mathbf{x}, t)}{\partial t} = -\frac{\delta H}{\delta \phi(\mathbf{x}, t)} + \boldsymbol{\eta}(\mathbf{x}, t), \quad (2.2)$$

where $\boldsymbol{\eta}(\mathbf{x}, t)$ is a random Gaussian variable with mean zero and variance given by $\langle \boldsymbol{\eta}(\mathbf{x}, t) \boldsymbol{\eta}(\mathbf{x}', t') \rangle = 2T \delta(t - t') \delta(\mathbf{x} - \mathbf{x}')$.

For conserved fields, we add $-\nabla^2$ in front of the r.h.s to get the Cahn-Hilliard equation

$$\frac{\partial \phi(\mathbf{x}, t)}{\partial t} = \nabla^2 \left(\frac{\delta H}{\delta \phi(\mathbf{x}, t)} \right) + \boldsymbol{\eta}(\mathbf{x}, t), \quad (2.3)$$

where the variance of $\boldsymbol{\eta}(\mathbf{x}, t)$ is $\langle \boldsymbol{\eta}(\mathbf{x}, t) \boldsymbol{\eta}(\mathbf{x}', t') \rangle = -2T \delta(t - t') \nabla^2 \delta(\mathbf{x} - \mathbf{x}')$.

We shall see below that the limit $n \rightarrow \infty$ allows to solve the dynamics in both cases. The key point that makes the model exactly soluble is that in the limit of $n \rightarrow \infty$, the replacement $\phi^2/n \rightarrow \langle \phi^2 \rangle$, where ϕ is one of the components of $\boldsymbol{\phi}$, can be made. The two types of dynamics are now successively considered.

III. NON-CONSERVED ORDER PARAMETER: SIMPLE AGING

The time dependent Ginzburg-Landau equation (2.2) associated to the Hamiltonian (2.1) is

$$\frac{\partial \phi}{\partial t} = \nabla^2 \phi + \phi - \frac{1}{n} (\phi^2) \phi + \boldsymbol{\eta}, \quad (3.1)$$

where the dependence on space and time has been removed for clarity. This differential equation is associated with random initial conditions, in order to reproduce the quench experiment described in the introduction, and $\phi(\mathbf{x}, 0)$ is taken from a Gaussian distribution with zero mean and variance $\langle \phi(\mathbf{x}, 0) \phi(\mathbf{x}', 0) \rangle = \Delta \delta(\mathbf{x} - \mathbf{x}')$. From now on, we work at $T = 0$. In the coarsening problem, temperature does not play an essential role, provided it is below the critical temperature. The scaling regime can then be directly studied at $T = 0$. The review paper [2] provides a longer discussion of that point, and we discuss below how our results may be (slightly) changed by a non-zero temperature.

The large- n limit results in the following equations which have to be self-consistently solved:

$$\frac{\partial \phi}{\partial t} = \nabla^2 \phi + a(t) \phi; \quad a(t) = 1 - \langle \phi^2 \rangle. \quad (3.2)$$

The solution is discussed in Refs. [2,13], and one finds for the Fourier transform $\phi(\mathbf{k}, t) = \int d^d \mathbf{x} \phi(\mathbf{x}, t) e^{-i\mathbf{k} \cdot \mathbf{x}}$

$$\phi(\mathbf{k}, t) = \phi(\mathbf{k}, 0) e^{-k^2 t} \left(\frac{t}{t_0} \right)^{d/4}, \quad (3.3)$$

where $t_0 \equiv \Delta^{2/d}/8\pi$. It is now easy to compute the structure factor

$$S(\mathbf{k}, t) \equiv \frac{1}{V} \langle \phi(\mathbf{k}, t) \phi(-\mathbf{k}, t) \rangle = (8\pi t)^{d/2} e^{-2k^2 t}. \quad (3.4)$$

We used $\langle \phi(\mathbf{k}, 0) \phi(-\mathbf{k}, 0) \rangle = \Delta V$ from initial conditions. The structure factor may be written as $S(\mathbf{k}, t) = L^d g(kL)$, with $L(t) = t^{1/2}$ and $g(x) = (8\pi)^{d/2} \exp(-2x^2)$, demonstrating the validity of the scaling hypothesis in that case.

The autocorrelation function is defined as

$$C(t, t_w) \equiv \frac{1}{V} \int d^d \mathbf{x} \langle \phi(\mathbf{x}, t) \phi(\mathbf{x}, t_w) \rangle = \frac{1}{V} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \langle \phi(\mathbf{k}, t) \phi(-\mathbf{k}, t_w) \rangle \quad (3.5)$$

and may be easily computed:

$$C(t, t_w) = \left[\frac{2\sqrt{tt_w}}{t + t_w} \right]^{d/2}. \quad (3.6)$$

Defining the scaling variable $\lambda_1 \equiv t/t_w$, $C(t, t_w)$ can be rewritten

$$C(t, t_w) = F_1(\lambda_1); \quad F_1(x) \equiv \left[\frac{2\sqrt{x}}{1+x} \right]^{d/2}. \quad (3.7)$$

This last equation means that the autocorrelation function exhibits a simple aging behaviour. We have then illustrated on a concrete model the scaling approach to domain growth described in the introduction. We shall see in the next section the differences arising when sub-aging is present.

Let us note here that a finite temperature does not affect the above discussion, since it simply introduces a short-time relaxation in the correlation function, that does not depend on the waiting time t_w and corresponds to an *equilibrium relaxation inside the growing domains*. The long-time relaxation we are interested in, and which corresponds to the growth of the domains themselves is still described by (3.7).

IV. CONSERVED ORDER PARAMETER: SUB-AGING

The Cahn-Hilliard equation (2.3) associated to the Hamiltonian (2.1) is given by (still at $T = 0$)

$$\frac{\partial \phi}{\partial t} = -\nabla^2 \left[\nabla^2 \phi + \phi - \frac{1}{n} (\phi^2) \phi \right], \quad (4.1)$$

and is solved following the same steps as previously, leading to [2,13]:

$$\phi(\mathbf{k}, t) = \phi(\mathbf{k}, 0) \exp \left(-k^4 t + k^2 \sqrt{\frac{dt}{2}} \ln \left(\frac{t}{t_0} \right) \right), \quad (4.2)$$

with $t_0 \equiv \Delta^{4/d}/(16\pi)^2$. The structure factor reads in that case

$$S(\mathbf{k}, t) \sim [L_1(t)^d]^{f(kL_2(t))}, \quad (4.3)$$

where $f(x) \equiv 2x^2 - x^4$. In this expression, two characteristic length scales have been defined: $L_1(t) \equiv t^{1/4}$, and $L_2(t) \equiv (\frac{8t}{d \ln(t/t_0)})^{1/4}$. In the standard scaling form, $S(\mathbf{k}, t) \sim L^d g(kL)$, the structure factor varies as L^d with a prefactor depending on the scaling variable kL , whereas for

the multiscaling form (4.3), S varies as L_1^α , with an exponent α which depends continuously on the scaling variable kL_2 . The two scalings are thus completely different.

Coniglio and Zannetti [13] have interpreted this multiscaling in terms of domains composed of sub-domains, each sub-domain growing at a different rate. The initial motivation for the present work was indeed to investigate the possible existence of a ‘hierarchy’ of time scales, similar to the one found in mean-field spin glass models (‘ultrametricity in time’) [1,10,16]. A different effect arises instead. Using eq.(4.2), one easily gets for the autocorrelation function

$$C(t, t_w) \sim \frac{1}{(t + t_w)^{d/4}} \exp \left(\frac{d}{8} \frac{\left(\sqrt{t \ln(t/t_0)} + \sqrt{t_w \ln(t_w/t_0)} \right)^2}{t + t_w} \right). \quad (4.4)$$

It is obvious from this expression that $C(t, t_w)$ cannot be written as a function of t/t_w only. The physical key ingredient for the absence of the usual scaling is the presence of two different length scales in the system.

We prove now analytically that eq.(4.4) implies sub-aging. It has to be remarked first that when the time difference $\tau \equiv t - t_w$ is equal to t_w , one has

$$C(t_w + t_w, t_w) \underset{t_w \rightarrow \infty}{\sim} \frac{1}{t_w^{(3-2\sqrt{2})d/24}} \rightarrow 0. \quad (4.5)$$

In the asymptotic limit of large waiting times, the relaxation of $C(t, t_w)$ is complete in times $\tau \ll t_w$. In that regime, one can show that

$$C(t, t_w) \underset{\tau \ll t_w}{\sim} \exp \left(-\frac{d \ln t_w}{64} \left(\frac{\tau}{t_w} \right)^2 \right). \quad (4.6)$$

Defining the scaling variable $\lambda_2 \equiv \tau \sqrt{\ln t_w} / t_w$, eq.(4.4) can finally be rewritten

$$C(t, t_w) \sim F_2(\lambda_2); \quad F_2(x) \equiv \exp \left(-\frac{dx^2}{64} \right). \quad (4.7)$$

The relaxation time grows hence as $t_r \sim t_w / \sqrt{\ln t_w}$, *i.e.* more slowly than t_w : *this is a sub-aging behaviour*. It is moreover possible to compute the function $h(t)$ discussed in the introduction. The scaling form $C(t, t_w) = \mathcal{C} \left(\frac{h(t)}{h(t_w)} \right)$ should be valid in the two-time regime where both times are large, but with a non-zero value of the correlation function. In the present case, this regime is characterized by

$$t_w \rightarrow \infty, \quad \tau \sim \frac{t_w}{\sqrt{\ln t_w}}. \quad (4.8)$$

We have seen that a natural choice for $h(t)$ would be $L_1(t)$ or $L_2(t)$, *i.e.* a length scale, since it is a common interpretation. *This does not work*, and a more complicated form has to be found. It is straightforward to realize that a possible choice is an enhanced power law:

$$\mathcal{C}(x) = \exp \left(-\frac{d}{288} \ln^2(x) \right); \quad h(t) = \exp \left((\ln t)^{3/2} \right). \quad (4.9)$$

The function h is neither L_1 nor L_2 , but a combination of the two, and therefore does not have a direct physical interpretation: $h(t) \sim \exp \left((L_1/L_2)^6 \right)$.

V. RESPONSE FUNCTIONS: INFINITE EFFECTIVE TEMPERATURES

It is also relevant to study the response functions for aging systems, since it is a major prediction of the dynamical mean-field theory for spin glasses that interesting informations are encoded in the susceptibilities [1,17]. Up to now, we have studied aging in the two-time correlation functions $C(t, t_w)$. In glassy systems, aging is also found in the related response functions $R(t, t_w)$, associated with a breakdown of the fluctuation dissipation theorem which at equilibrium would be $TR(t, t_w) = \partial C(t, t_w)/\partial t_w$. This is taken into account by introducing an effective temperature T_{eff} through [17]

$$T_{\text{eff}}(q) = \lim_{t_w \rightarrow \infty} \left. \frac{\frac{\partial C(t, t_w)}{\partial t_w}}{R(t, t_w)} \right|_{C(t, t_w)=q}. \quad (5.1)$$

In coarsening systems, however, response functions have been shown numerically and analytically to be weak, in the sense that $T_{\text{eff}} \rightarrow \infty$ [18,19]. This property has been related to the decreasing density of topological defects (domain walls) during the coarsening. In the case of the $O(n)$ model, no topological defects are present if $n > d$, which is naturally the case in the large- n limit. We compute then $R(t, t_w)$ in the both cases studied above to obtain T_{eff} . We refer the reader to ref. [19] for the method, since we follow exactly the same steps. We get the two following expressions:

$$R(t, t_w) \sim \left(\frac{t}{t_w} \right)^{d/4} \left(\frac{1}{t - t_w} \right)^{d/2}, \quad (5.2)$$

in the non-conserved case, and

$$R(t, t_w) \sim \frac{1}{(t - t_w)^{(d+2)/4}} \exp \left(\frac{d}{8} \frac{(\sqrt{t \ln t} - \sqrt{t_w \ln t_w})^2}{t - t_w} \right), \quad (5.3)$$

in the conserved case (we dropped out all numerical constants). Combining eqs.(3.6,4.4,5.2,5.3), it is easy to show that for the non-conserved and the conserved case successively, one has:

$$T_{\text{eff}}(q) \sim \lim_{t_w \rightarrow \infty} t_w^{d/2-1}, \quad T_{\text{eff}}(q) \sim \lim_{t_w \rightarrow \infty} \frac{t_w^{(d-2)/4}}{(\ln t_w)^{(d+2)/8} \exp(\sqrt{\ln t_w})}. \quad (5.4)$$

This holds for $0 < q < 1$, and shows that for $d > 2$, although there is no interpretation here in terms of defects, the effective temperature is infinite, as has been found so far in all domain growth processes [4,18,19].

We studied in this letter the aging dynamics of the $O(n)$ model in the large- n limit. We showed that when the order parameter is not conserved, standard scaling laws hold, leading to a simple aging behaviour. We investigated the more interesting case of a conserved dynamics, and were able to show that the multiscaling observed in the structure factor does not imply a hierarchy of time scales ('ultrametricity in time' [16]). Rather, the relaxation takes place in a time scale which is shorter than the waiting time, $t_r \sim t_w / \ln^{a-1}(t_w)$ with $a = 3/2$, the correlation function being well represented in that regime by $C(t, t_w) = \mathcal{C}(h(t)/h(t_w))$, where h is an enhanced power law $h(t) = \exp(\ln^a(t))$. This simple example exhibits then a very rich aging behaviour, whose origin is the presence of two different length scales during the coarsening process. It shows also that the interpretation of $h(t)$ as a length scale may in some cases be

misleading. The enhanced exponential form that has been successfully used to fit spin glass experiments arises naturally from our computation. It implies that the relaxation time scales as $t_r \sim t_w / \ln^{a-1}(t_w)$, which could hardly be experimentally distinguishable from a power law $t_r \sim t_w^\mu$, when μ is very near to one, as it is in spin glasses.

ACKNOWLEDGMENTS

I sincerely thank J. KURCHAN who suggested and followed this work, J.-L. BARRAT and J.-PH. BOUCHAUD for their interest and encouragements, L. F. CUGLIANDOLO and M. SELLITTO for their help during the preparation of the manuscript.

-
- [1] Bouchaud J.-Ph., Cugliandolo L. F., Kurchan J. and Mézard M., *Spin Glasses and Random Fields*, edited by Young A. P. (World Scientific, Singapore) 1998; Bouchaud J.-Ph., Lectures notes for the “*Soft and fragile matter*” Summer School, St Andrews 1999, preprint cond-mat/9910387.
 - [2] Bray A. J., Adv. in Phys. **43** (1994) 357.
 - [3] Fisher D. S., Physica D **107** (1997) 204; Fisher D. S. and Huse D. A., Phys. Rev. B **38** (1998) 373.
 - [4] Shukla P. and Singh S., Phys. Rev. B **23** (1981) 4661; Ciuchi S. and de Pasquale F., Nucl. Phys. B **300** (1988) 31; Cugliandolo L. F. and Dean D. S., J. Phys. A **28** (1995) 4213; Godrèche C. and Luck J.-M., J. Phys. A **30** (1997) 6245; Barrat A., Burioni R. and Mézard M., J. Phys. A **29** (1996) 1331; Cugliandolo L. F., Kurchan J. and Parisi G., J. Phys. A **27** (1994) 5749; Zippold W., Kuehn R. and Horner H., preprint cond-mat/9904329.
 - [5] Kisker J., Santen L., Schreckenberg M. and Rieger H., Phys. Rev. B **53** (1996) 6418.
 - [6] Vincent E., Hammann J., Ocio M., Bouchaud J.-Ph. and Cugliandolo L. F., *Complex behaviour of glassy systems*, edited by Rubi M. (Springer Verlag, Berlin) 1997.
 - [7] Struik L. C. E., *Physical aging in amorphous polymers and other materials* (Elsevier, Amsterdam) 1978.
 - [8] Kob W. and Barrat J.-L., Phys. Rev. Lett. **78** (1997) 4581.
 - [9] Cippelletti L., Manley S., Ball R. C. and Weitz D. A., Phys. Rev. Lett. **84** (2000) 2275.
 - [10] Rinn B., Maass P. and Bouchaud J. Ph., preprint cond-mat/0001161.
 - [11] Laloux L. and Le Doussal P., Phys. Rev. E **57** (1998) 6296.
 - [12] Raising the problem of finite size effects in spin glass experiments, a possible explanation for sub-aging has been given in: Joh Y. G., Orbach R., Wood G. G., Hammann J. and Vincent E., preprint cond-mat/0002040; Bouchaud J.-Ph., Vincent E. and Hammann, J. Phys. I (France) **4** (1994) 139.
 - [13] Coniglio A. and Zannetti M., Europhys. Lett. **10** (1989) 575.
 - [14] Bray A. J., Humayun K., Phys. Rev. Lett. **68** (1992) 1559.
 - [15] Castellano C. and Zannetti M. cond-mat/9906094; Castellano C. and Zannetti M., Phys. Rev. E **58** (1998) 5410.
 - [16] Cugliandolo L. F. and Kurchan J., J. Phys. A **27** (1994) 5749.
 - [17] Cugliandolo L. F., Kurchan J. and Peliti L., Phys. Rev. E **55** (1997) 3898.
 - [18] Barrat A., Phys. Rev. E **57** (1998) 3629.
 - [19] Berthier L., Barrat J.-L. and Kurchan J., Eur. Phys. J. B **11** (1999) 635.